LXXIV. Spherical Trigonometry *reduced to* Plane, *by* Francis Blake, *Esq. F. R. S.*

Read May 7, 1752.

IT is obſervable, that the analogies of ſpherical trigonometry, excluſive of the terms co-ſine and co-tangent, are applicable to plane, by only changing the expreſſion, ſine or tangent of ſide, into the ſingle word, ſide *: ſo that the buſineſs of plane trigonometry, like a corollary to the other, is thence to be inferr'd. And the reaſon of this is obvious; for analogies raiſed not only from the conſideration of a triangular figure, but the curvature alſo, are of conſequence more general; and tho' the latter ſhould be held evaneſcent by a diminution of the ſurface, yet what depends upon the triangle, will nevertheleſs remain. Theſe things may have been obſerved, I ſay; but upon reviſing the ſubject, it further occurr'd to me, and I take it to be new, that from the axioms of only plane trigonometry, and almoſt independent of ſolids, and the doctrine of the ſphere, the ſpherical caſes are likewiſe to be ſolved.

Suppoſe, firſt, that the three ſides of a ſpherical triangle, *a b d* (Fig. 1.) are given to find an angle, *a*; which caſe will lay open the method, and lead on to the other caſes, in a way, that to me appears the moſt natural. It is allow'd, that the tangents, *a e*, *a f*, of the ſides, *a d*, *a b*, including an angle, *a*, make a plane angle equal to it; and it is evident, that the other ſide, *d b*, determines the angle made by the ſecants *c e*, *c f*, at *c* the centre of the ſphere; whence the diſtance, *e f*, betwixt the tops of thoſe ſecants, is

Kkk given

* See M. De la Caille's remark at the end of the ſpherical trigonometry prefix'd to his Elements of Aſtronomy.

given by case the fifth of oblique plane triangles (see *Heynes's Trigonom.*) which, with the aforesaid tangents, reduces it to case the 6th of oblique plane triangles also *: and thus this 11th case of oblique triangles, so intricate hitherto, becomes perfectly easy. The 12th case is reducible to the 11th, and the rest, whether right-angled, or oblique, we are authorised to look upon as reducible to right-angled triangles, whose sides are not quadrants, but either greater or less than such. Conceive therefore, now, in a right-angled spherical triangle, *gkh* (Fig. 2.) that the tangent, *g m*, and secant, *e m*, of either leg, *g k*, is already drawn; and in the point, *m*, of their union, draw a perpendicular, *m l*, to *e m*, the secant, directly above the other leg, *viz.* a perpendicular to the plane of the secant and tangent, that it may be perpendicular to both (*Eucl.* 4, 11); for then will the tangent, *g l*, of the hypothenuse, *g h*, drawn from the same point, which that of the leg was, constantly terminate in the perpendicular line, that the radius and tangent may make a right-angle (*Eucl.* 18, 3). Whence these tangents, *g m*, *g l*, and the perpendicular line, *m l*, together with the secants, *c m*, *c l*, will evidently form two right-angled plane triangles, *g m l*, *c m l*; and to one or other of these the spherical cases are easily transferr'd. Thus, if in the spherical triangle, *g k h*, the hypothenuse, *g h*, base, *g k*, and angle, *g*, at the base, be the parts given and required, when any two are given, the third

* The angle to be found in this case must always be that formed by the two tangents.

third may be determined by means of a plane triangle; and at a ſingle operation. We have, for inſtance, in the right-angled plane triangle, *g m l*, formed as above, the baſe, *g m*, and hypothenuſe, *g l*, to find, by caſe the 5th of right-angled plane triangles, the angle included, which is the ſame as on the ſphere. And then if the baſe, *g k*, the angle, *g*, at the baſe, and perpendicular, *k h*, be the ſpherical parts given and required; or if the angles, *g* and *h*, and the hypothenuſe, *g h*, be the parts given and required, we have only that former proportion of the hypothenuſe and baſe, and angle at the baſe, in the triangles, *PND*, *DFG*, obtained by the complements, to transfer to the plane. But ſecondly, ſuppoſe the ſpherical proportion is of the three ſides, any two being given, the third may be alſo found at a ſingle operation, in the ſecond right-angled plane triangle, *c m l*, form'd as above. We have, for inſtance, the hypothenuſe and baſe, *c l*, *c m*, *viz.* the ſecant of the ſpherical hypothenuſe and baſe *g h*, *g k*, to find, by the 5th of right-angled plane triangles, the angle, *c*, at the center, which is the meaſure of *k h*, the ſide that was ſought. And then again, if the hypothenuſe, one leg, and the oppoſite angle be the ſpherical parts given and required; or if the two angles and a leg be the parts given and required, we have only the former proportion of the three ſides in the triangles, *PND*, *DFG*, obtained by the complements, to transfer to the plane. Whence, the ſix proportions of right-angled ſpherical triangles being comprehended in this method, it is fully demonſtrated, that all the caſes of theſe triangles are ſo to be reſolved.

The ſame might be deduced without the method of complements, but neither in ſo ſhort nor ſatisfactory a way, and it ſhall therefore be omitted. I have communicated this upon account of its perſpicuity, and ſuppoſing, that in an age ſo greatly advanced in mathematical learning, the leaſt hint of what is new would not be unacceptable.

Fig. 1.

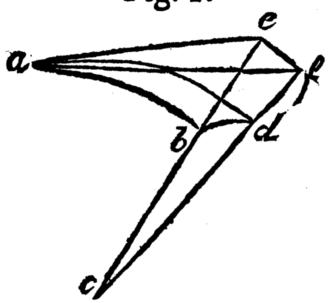


Fig. 2.

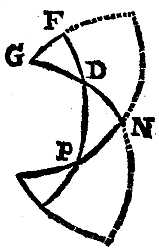
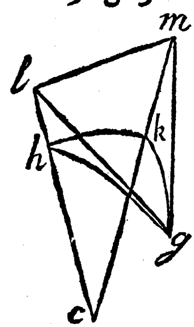


Fig. 3.



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